RING-THEORETIC PROPERTIES OF PvMDS

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Abstract. We extend to Prüfer v-multiplication domains some distinguished ring-theoretic properties of Prüfer domains. In particular we consider the $t\#\#$-property, the t-radical trace property, w-divisoriality and w-stability.

Introduction

A Prüfer v-multiplication domain, for short a PvMD, is a domain whose localizations at t-maximal ideals are valuation domains [22]. For this reason, the ideal-theoretic properties of valuation domains globalize to t-ideals of PvMDs and several properties of ideals of Prüfer domains hold for t-ideals of PvMDs: for example a domain is a PvMD if and only if each t-finite t-ideal is t-invertible. The aim of this paper is to show that, introducing suitable t-analogues of some distinguished properties of integral domains, Prüfer domains and PvMDs have a similar behaviour also from the ring-theoretic point of view. We recall that the class of PvMDs, besides Prüfer domains, includes Krull domains and GCD-domains.

As a matter of fact, the t-operation is not always as good as the w-operation for extending certain properties that hold in the classical case, that is in the d-operation setting. Thus in general it is often more convenient to consider the w-analogue of a given property (see for instance [43, 44, 9]). However in a PvMD the w-operation and the t-operation coincide [30] and one can use indifferently these two star operations.

In Section 1, we deal with the $t\#\#$-property and the $tRTP$-property. PvMDs satisfying the $t\#\#$-property have been studied in [15, Section 2]. Here we characterize PvMDs with the $tRTP$-property; getting for example that a PvMD is a $tRTP$-domain if and only if each v-finite divisorial ideal has at most finitely many minimal primes. Then, generalizing the Prüfer case, we show that the $t\#\#$-property and the $tRTP$-property are strictly connected for PvMDs. Among other results, we prove that a PvMD satisfying the $t\#\#$-property is a $tRTP$-domain and that the converse holds if each t-prime is branched. We also show that an almost Krull domain satisfying the $t\#\#$-property is a Krull domain.

In Section 2 we introduce the notion of w-stability and relate it to w-divisoriality, a property defined and studied in [9]. First we show that the study of w-stability can be reduced to the t-local case. Then we use this result to extend to PvMDs some properties of stable and divisorial Prüfer domains. For example, we prove that w-stability of t-primes enforces a PvMD to be a generalized Krull domains and that an integrally closed w-stable domain is precisely a generalized Krull domain with t-finite character. We also characterize w-stable w-divisorial PvMDs and show that these domains behave like totally divisorial Prüfer domains.

We assume that the reader is familiar with the language of star operations [18, Sections 32 and 34]. We recall some definitions and basic properties.

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Throughout this paper R will denote an integral domain with quotient field K and we will assume that $R \neq K$.

A *star operation* is a map $I \to I^*$ from the set $F(R)$ of nonzero fractional ideals of R to itself such that:

(1) $R^* = R$ and $(aI)^* = aI^*$, for all $a \in K \setminus \{0\}$;

(2) $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$;

(3) $I^{**} = I^*$.

A star operation $*$ is of *finite type* if $I^* = \bigcup\{J^*\,;\, J \subseteq I$ and J is finitely generated$\}$, for each $I \in F(R)$. To any star operation $*$, we can associate a star operation $*_f$ of finite type by defining $I^{*_f} = \bigcup J^*$, with the union taken over all finitely generated ideals J contained in I. Clearly $I^{*_f} \subseteq I^*$.

An ideal $I \in F(R)$ is a $*$-*ideal* if $I = I^*$ and is $*$-*finite* if $I^* = J^*$ for some finitely generated ideal J. A $*$-finite $*$-ideal is also called a $*$-*ideal of finite type*.

A $*$-*prime* is a prime $*$-ideal and a $*$-*maximal* ideal is an ideal that is maximal in the set of the proper $*$-ideals. A $*$-maximal ideal (if it exists) is a prime ideal. If $*$ is a star operation of finite type, an easy application of Zorn's Lemma shows that the set $*$-Max(R) of the $*$-maximal ideals of R is not empty. In this case, for each $I \in F(R)$, $I^* = \bigcap_{M\in *\text{-Max}(R)} I^*R_M$; in particular $R = \bigcap_{M\in *\text{-Max}(R)} R_M$ [22].

For any star operation $*$, the set of $*$-ideals of R is a semigroup under the $*$-*multiplication*, defined by $(I, J) \mapsto (IJ)^*$, with unity R. An ideal $I \in F(R)$ is called $*$-*invertible* if I^* is invertible with respect to the $*$-multiplication. In this case the $*$-*inverse* of I is $(R : I)$. Thus I is $*$-invertible if and only if $(I(R : I))^* = R$.

The identity is a star operation, called the *d-operation*. The *v-operation* (or *divisorial closure*), the *t-operation* and the *w-operation* are the best known nontrivial star operations and are defined in the following way. For each $I \in F(R)$, we set $I_v := (R\colon (R\colon I))$ and $I_t := \bigcup J_v$ with the union taken over all finitely generated ideals J contained in I. Hence the t-operation is the finite type star operation associated to the v-operation. The w-operation is the star operation of finite type defined by setting $I_w := \bigcap_{M\in t\text{-Max}(R)} IR_M$. We have w-Max$(R) = t$-Max(R) and $IR_M = I_wR_M \subseteq I_tR_M$, for each $M \in t$-Max(R). Thus $I_w \subseteq I_t \subseteq I_v$. Also, an ideal $I \in F(R)$ is w-invertible if and only if it is t-invertible.

A Prüfer domain is an integrally closed domain such that $d = t$ [18, Proposition 34.12] and a PvMD is an integrally closed domain such that $w = t$ [30, Theorem 3.1].

The v-, t- and w-operations on R can be extended to the set of nonzero R-submodules of K by setting $E^v := (R : (R : E))$, $E_t = \bigcup\{F_v;\, F \subseteq E\,; F$ finitely generated$\}$ and $E_w = \bigcap\{E_M;\, M \in t$-Max$(R)\}$, for each non zero R-submodule E of K. In this way, one obtains *semistar operations* on R. For more details, see for example [12]. By viewing w as a semistar operation on R, we can say that an overring T of a domain R is *t-linked* over R if $T_w = T$ [4, Proposition 2.13]. Each overring of R is t-linked precisely when $d = w$ [4, Theorem 2.6].

We denote by t-Spec(R) the set of t-prime ideals of R. Each height-one prime is a t-prime and each prime minimal over a t-ideal is a t-prime. We say that R has *t-dimension one* if each t-prime ideal has height one.

We now define the ring-theoretic properties considered in this paper.

The t##-property. The #-property and the ##-property were introduced by R. Gilmer [19] and R. Gilmer and W. Heinzer [20] respectively. A domain R has the *#-property*, or it is a *#-domain*, if $\bigcap_{M\in\mathcal{M}_1} R_M \neq \bigcap_{M\in\mathcal{M}_2} R_M$, for any pair of distinct nonempty sets $\mathcal{M}_1$ and $\mathcal{M}_2$ of maximal ideals. Any overring of a one-dimensional Prüfer #-domain is a #-domain [19, Corollary 2], but in general

the #-property is not inherited by overrings [20, Section 2]. One says that R has the *##-property*, or it is a *##-domain*, if each overring of R is a #-domain.

The t#-property was introduced and studied in [15]. A domain R has the *t#-property* (or is a *t#-domain*) if $\bigcap_{M\in\mathcal{M}_1} R_M \neq \bigcap_{M\in\mathcal{M}_2} R_M$ for any two distinct subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ of t-Max(R). Although in [15] it was explored the transfer of the t#-property to some distinguished classes of overrings, it was not given any definition for the t##-property. Here, we say that R has the *t##-property* (or is a *t##-domain*) if $\bigcap_{P\in\mathcal{P}_1} R_P \neq \bigcap_{P\in\mathcal{P}_2} R_P$ for any two distinct subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ of pairwise incomparable t-prime ideals of R. Our definition is motivated by the fact that for a PvMD this is equivalent to say that each t-linked overring has the t#-property [15, Proposition 2.10].

The tRTP-property. If R is an integral domain and M is a unitary R-module, the *trace* of M is the ideal of R generated by the set $\{f(m)\,; f \in \mathrm{Hom}_R(M,R)\,, m \in M\}$. An ideal J of R is called a *trace ideal* or a *strong ideal* if it is the trace of some R-module M. This happens if and only if $J = I(R:I)$, for some nonzero ideal I of R, equivalently $(J:J) = (R:J)$ [10, Lemmas 4.2.2. and 4.2.3]. If V is a valuation domain, a trace ideal is either equal to V or it is prime [10, Proposition 4.2.1]; this fact led to the consideration of several conditions related to trace ideals [28]. The radical trace property was introduced by W. Heinzer and I. Papick [24] and is particularly significant for Prüfer domains [24, 33]. R is a domain satisfying the *radical trace property*, or it is an *RTP-domain*, if each proper strong ideal is a radical ideal, that is, for each nonzero ideal I of R, either $I(R:I) = R$ or $I(R:I)$ is a radical ideal.

A. Mimouni studied trace properties in the setting of star operations, in particular he considered the t-operation [35]. As in [35], we say that a domain R has the *t-radical trace property*, or it is a *tRTP-domain*, if each proper strong t-ideal of R is a radical ideal. This is equivalent to say that, for each nonzero ideal I of R, either $(I(R:I))_t = R$ or $(I(R:I))_t$ is a radical ideal.

w-divisoriality. The class of domains in which each nonzero ideal is divisorial has been studied, independently and with different methods, by H. Bass [2], E. Matlis [34] and W. Heinzer [23] in the sixties. Following S. Bazzoni and L. Salce [3], a domain in which each nonzero ideal is divisorial is now called a *divisorial domain* and a domain such that each overring is divisorial is called *totally divisorial*.

The most suitable star analogue of divisoriality is the notion of w-divisoriality that was introduced and extensively studied in [9]. A *w-divisorial domain* is a domain such that each w-ideal is divisorial.

w-stability. Motivated by earlier work of H. Bass [2] an J. Lipman [32] on the number of generators of an ideal, in 1974 J. Sally and W. Vasconcelos defined a Noetherian ring R to be *stable* if each nonzero ideal of R is projective over its endomorphism ring $\mathrm{End}_R(I)$ [41, 42]. When I is a nonzero ideal of a domain R, then $\mathrm{End}_R(I) = (I:I)$; thus a domain R is stable if each nonzero ideal I of R is invertible in the overring $(I:I)$. B. Olberding showed that stability and divisoriality are strictly connected and that stability is particularly significant in the context of Prüfer domains [10, 36, 37, 38, 39].

We introduce the notion of w-stability in Section 2. We say that a w-ideal I of a domain R is *w-stable* if I is w-invertible in the overring $E(I) := (I:I)$, that is $(I(E(I):I))_w = E(I)$, and say that R is *w-stable* if each w-ideal of R is w-stable. For a more general notion of stability with respect to a semistar operation we refer the reader to the forthcoming paper [16].

1. t##-Property and t-radical trace property

The ##-property and the radical trace property are closely related for a Prüfer domain. In this section we compare the t-analogues of these two properties for PvMDs.

Several characterizations of PvMDs satisfying the $t\#\#$-property have been given in [15, Section 2]. For the study of the $tRTP$-property, we need some results on branched t-primes. Recall that a prime ideal P of a domain R is *branched* if there exists a P-primary ideal distinct from P. Clearly P is branched if and only if PR_P is branched. Since the localization of a PvMD at a t-prime is a valuation domain, the branched t-primes of PvMDs can be characterized by properties similar to those well known for the branched primes of Prüfer domains [18, Theorem 23.3 (e)].

Lemma 1.1. *Let R be a PvMD and $J := x_1R + \cdots + x_nR$ a nonzero finitely generated ideal such that $J_v \neq R$. If P is a t-prime containing J, then P is minimal over J_v if and only if P is minimal over J, if and only if P is minimal over x_iR, for some i, $1 \leq i \leq n$.*

Proof. It is enough to observe that, since R_P is a valuation domain, we have $J_vR_P = J_tR_P = JR_P = x_iR_P$, for some i, $1 \leq i \leq n$. □

Proposition 1.2. *Let R be a PvMD and P a t-prime of R. The following conditions are equivalent:*

(i) *P is branched;*
(ii) *P is a minimal prime of a principal ideal;*
(iii) *P is a minimal prime of a finitely generated ideal;*
(iv) *P is a minimal prime of a v-finite divisorial ideal;*
(v) *P is not the union of the set of (t)-primes of R properly contained in P.*

Proof. The equivalence of conditions (i), (ii) and (v) is obtained by localizing at P and using [18, Theorem 17.3 (e)]. The equivalence of conditions (ii), (iii) and (iv) follows from Lemma 1.1. □

In any commutative ring with unity, if each minimal prime of an ideal I is the radical of a finitely generated ideal, then I has only finitely many minimal primes [21, Theorem 1.6]. By passing through the t-Nagata ring, we now show that a similar result holds for t-ideals of PvMDs.

If R is an integral domain, we set $R(X) := R[X]_N$, where $N = \{f \in R[X] : c(f) = R\}$ and $R\langle X\rangle := R[X]_{N_t}$, where $N_t = \{f \in R[X];\, c(f)_t = R\}$. $R(X)$ is called the *Nagata ring* of R and $R\langle X\rangle$ the *t-Nagata ring of R* [12, 29, 30].

B. G. Kang proved that R is a PvMD if and only if $R\langle X\rangle$ is a Prüfer (indeed a Bezout) domain [30, Theorem 3.7]. In addition, there is a lattice isomorphism between the lattice of t-ideals of R and the lattice of ideals of $R\langle X\rangle$ [30, Theorem 3.4]. More precisely, we have:

Proposition 1.3. *Let R be a PvMD. Then the map $I_t \mapsto IR\langle X\rangle$ is an order-preserving bijection between the set of t-ideals of R and the set of nonzero ideals of $R\langle X\rangle$, whose inverse is the map $J \mapsto J \cap R$. Moreover, P is a t-prime (respectively, t-maximal) ideal of R if and only if $PR\langle X\rangle$ is a prime (respectively, maximal) ideal of $R\langle X\rangle$ and we have $R\langle X\rangle_{PR\langle X\rangle} = R[X]_{PR[X]} = R_P(X)$.*

Proposition 1.4. *Let R be a PvMD and I a proper t-ideal of R. If each minimal prime of I is the radical of a v-finite divisorial ideal, then I has finitely many minimal t-primes.*

Proof. Each minimal prime of a t-ideal I is a t-ideal of R. By Proposition 1.3, the map $P \mapsto PR\langle X\rangle$ is a bijection between the set of minimal primes of I and the set of minimal primes of $IR\langle X\rangle$. Moreover if J is a nonzero finitely generated ideal of R such that $P = \mathrm{rad}(J_v)$, then $PR\langle X\rangle = \mathrm{rad}(JR\langle X\rangle)$. Hence each minimal prime of $IR\langle X\rangle$ is the radical of a finitely generated ideal. By [21, Theorem 1.6], $IR\langle X\rangle$ has finitely many minimal primes and the same holds for I. □

If T is an overring of R, the w-operation and the t-operation on R, viewed as semistar operations, induce two semistar operations of finite type on T, which here are still denoted by w and t respectively. If in addition T is t-linked over R, the w-operation is a star operation on T [8, Proposition 3.16]. Note that this star operation, being spectral and of finite type [12], is generally smaller than the w-operation on T, that we denote by w' to avoid confusion.

Proposition 1.5. *Let R be a PvMD and T a t-linked overring of R. Then T is a PvMD and $w = t = t' = w'$ on T, where w' and t' denote respectively the w-operation and the t-operation on T.*

Proof. When R is a PvMD also T is a PvMD [30, Theorem 3.8 and Corollary 3.9]. In addition, if R is a PvMD the two semistar operations w and t coincide [11, Theorem 3.1 $((i) \Rightarrow (vi))$]. Hence $w = t$ and $w' = t'$ as star operations on T.

We next show that $t = t'$ on T. Let I be a nonzero ideal of T. Clearly, $I_t \subseteq I_{t'}$. On the other hand, we have $I_{t'} = \bigcap\{IT_M\,;\, M \in t'\text{-Max}(T)\}$. Since $I_t = \bigcap\{I_tT_N\,;\, N \in t\text{-Max}(T)\}$ [22, Proposition 4], to show that $I_{t'} \subseteq I_t$ it suffices to show that $t\text{-Max}(T) \subseteq t'\text{-Max}(T)$.

If $N \in t\text{-Max}(T)$, we have that $(N \cap R)_t \subseteq N_t \cap R_t = N \cap R$. Hence $N \cap R$ is a t-prime of R and $T_N \supseteq R_{N\cap R}$ are valuation domains. It follows that N is a t'-prime of T. In addition N is t'-maximal because it is t-maximal and each t'-prime of T is also a t-prime. □

If I is a w-ideal of R, then it is easily shown that $E(I)_w = E(I)$. Thus $E(I)$ is a t-linked overring of R. It follows that, when R is a PvMD, by Proposition 1.5, $E(I)$ is a PvMD and $w = t = t' = w'$ on $E(I)$.

Proposition 1.6. *Let R be a PvMD. Then:*

(1) *If I is a strong t-ideal of R, then $E(I) = S \cap T$, where $S = \bigcap_{P\in\text{Min}(I)} R_P$ and $T := \bigcap_{M\in t\text{-Max}(R), M\not\supseteq I} R_M$.*
(2) *If P is a t-prime of R which is not t-invertible, then $E(P) = (R : P) = R_P \cap T$, where $T := \bigcap_{M\in t\text{-Max}(R), M\not\supseteq P} R_M$.*
(3) *If P is a t-prime of R which is not t-invertible and R satisfies the ascending chain condition on radical t-ideals, then P is t-maximal in $E(P)$.*

Proof. (1) follows from [27, Theorem 4.5].

(2) If P is not t-invertible, then P is strong by [26, Proposition 2.3 and Lemma 1.2]. Whence (2) follows from (1).

(3) By part (2), $E(P) = (R : P)$. Since $E(P)$ is t-linked over R, then $E(P)$ is t-flat on R (that is $E(P)_Q = R_{Q\cap R}$ for each t-prime ideal Q of $E(P)$) [31, Proposition 2.10] and P is a t-ideal of $E(P)$ (Proposition 1.5). Let Q be a t-prime of $E(P)$ properly containing P. By t-flatness we can write $Q = (P'E(P))_t$, where $P' = Q \cap R$ is a t-prime of R properly containing P [6, Proposition 2.4]. By the ascending chain condition on radical t-ideals, $P' = \text{rad}(J_t)$ for some finitely generated ideal J [6, Lemma 3.7]. Since $P \subsetneq P'$, by checking t-locally, we get that $P \subsetneq J_t$. We have $R = (J(R : J))_t \subseteq (J(R : P))_t = (JE(P))_t \subseteq (P'E(P))_t = Q$. A contradiction. Hence $P = Q$ and so P is a t-maximal ideal of $E(P)$. □

Lemma 1.7. *Let R be a PvMD satisfying the $tRTP$-property. If P is a branched t-prime of R which is not t-invertible, then $R_P \not\supseteq T$, where $T = \bigcap_{M\in t\text{-Max}(R), M\not\supseteq P} R_M$.*

Proof. If $R_P \supseteq T$ then, by Proposition 1.6 (2), $T = E(P)$. Let Q be a P-primary ideal of R. Since P is a t-ideal, we may assume that Q is a t-ideal. We have $QT \subseteq PT = P \subseteq R$ and so $QT \subseteq QR_P \cap R = Q$. Hence $QT = Q$. If M is a t-maximal ideal of R such that $P \not\subseteq M$, then $Q \not\subseteq M$. Thus $(R : Q) \subseteq R_M$ and it follows that $(R : Q) \subseteq T$. Hence $Q(R : Q) = Q$. Since R is a $tRTP$-domain, then we must have $Q = P$. It follows that P is not branched. □

Theorem 1.8. *Let R be a PvMD. The following conditions are equivalent:*

(i) *R is a $tRTP$-domain;*

(ii) *Each branched t-prime P contains a finitely generated ideal J such that $J \subseteq P$ and $J \nsubseteq M$, for each $M \in t\text{-Max}(R)$ not containing P;*
(iii) *Each branched t-prime is the radical of a v-finite divisorial ideal;*
(iv) *Each nonzero principal ideal has at most finitely many minimal (t)-primes;*
(v) *Each nonzero finitely generated ideal has at most finitely many minimal t-primes;*
(vi) *Each v-finite divisorial ideal has at most finitely many minimal $(t\text{-})$primes.*

Proof. $(i) \Rightarrow (ii)$. Let P be a branched t-prime of R. If P is t-invertible, then P is v-finite and there is nothing to prove. If P is not t-invertible, then $E(P) = (R : P) = R_P \cap T$, where $T = \bigcap_{M \in t\text{-Max}(R), M \nsupseteq P} R_M$ (Proposition 1.6 (2)). Since $R_P \nsupseteq T$ (Lemma 1.7), there exists a finitely generated ideal J such that $J \subseteq P$ and $J \nsubseteq M$, for each $M \in t\text{-Max}(R)$ not containing P [7, Lemma 3.6].

$(ii) \Rightarrow (iii)$ Let P be a branched t-prime of R and J as in the hypothesis. By Proposition 1.2, P is minimal over a finitely generated ideal H. Hence P is the radical of the v-finite divisorial ideal $(J + H)_v$.

$(iii) \Rightarrow (iv)$. Let $x \in R$ be a nonzero nonunit and let $\{P_\alpha\}$ be the set of minimal primes of xR. By Proposition 1.2 each P_α is branched. Hence by hypothesis each P_α is the radical of a v-finite divisorial ideal. It follows from Proposition 1.4 that $\{P_\alpha\}$ is a finite set.

$(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ follow from Lemma 1.1.

$(iv) \Rightarrow (iii)$. Let P be a branched t-prime. By Proposition 1.2, P is minimal over a v-finite divisorial ideal I. Since I has finitely many minimal primes, then P is the radical of a v-finite divisorial ideal by [15, Lemma 2.13].

$(iii) \Rightarrow (i)$. By [35, Theorem 15] it is enough to show that for each strong t-ideal I and each minimal prime P of I we have $IR_P = PR_P$.

Assume that $IR_P \subsetneq PR_P$. Then P is branched, because R_P is a valuation domain. Thus $P = \text{rad}(H_v)$ for some finitely generated ideal H. Let $a \in P$ be such that $IR_P \subsetneq aR_P \subseteq PR_P$ and set $J = H + aR$. Then P is the radical of J_v. By checking t-locally, we have that $I \subseteq J_v$. In fact, let $M \in t\text{-Max}(R)$. If $P \nsubseteq M$, then $IR_M \subseteq R_M = J_vR_M$. If $P \subseteq M$, then $IR_M \subseteq IR_P$. Hence $a \notin IR_M$ and $IR_M \subsetneq J_vR_M$. Since I is strong, by Proposition 1.6(1), $(R : I) = E(I) \subseteq R_P$. Whence $J(R : J) \subseteq P(R : I) \subseteq PR_P$ and so $J(R : J) \subseteq P$. A contradiction because J is t-invertible. □

Since in a Prüfer domain the t-operation is trivial, we get the following corollary, due to T. Lucas. The equivalence $(i) \Leftrightarrow (ii)$ is [33, Theorem 23], while $(i) \Leftrightarrow (iv)$ is, to our knowledge, unpublished.

Corollary 1.9. *Let R be a Prüfer domain. The following conditions are equivalent:*
(i) *R is a RTP-domain;*
(ii) *Each branched prime is the radical of a finitely generated ideal;*
(iii) *Each principal ideal has at most finitely many minimal primes;*
(iv) *Each finitely generated ideal has at most finitely many minimal primes.*

The following theorem was stated for Prüfer domains in [33, Corollaries 25 and 26].

Theorem 1.10. *Let R be a PvMD.*
(1) *If R is a $t\#\#$-domain, then R is a $tRTP$-domain.*
(2) *If R is a $tRTP$-domain and each t-prime is branched, then R is a $t\#\#$-domain.*

Proof. A PvMD R has the $t\#\#$-property if and only if, for each t-prime ideal P, there exists a finitely generated ideal $J \subseteq P$ such that each t-maximal ideal containing J must contain P [15, Proposition 2.8]. Hence we can apply Theorem 1.8, $(i) \Leftrightarrow (ii)$. □

Theorem 1.11. *Let R be a PvMD. The following conditions are equivalent:*

(i) *R satisfies the ascending chain condition on radical t-ideals;*
(ii) *R is a $tRTP$-domain satisfying the ascending chain condition on prime t-ideals;*
(iii) *R is a $t\#\#$-domain satisfying the ascending chain condition on prime t-ideals;*
(iv) *R is a $t\#\#$-domain and each t-prime is branched.*

Proof. $(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$ by [15, Proposition 2.14].

$(ii) \Leftrightarrow (iii)$ By Proposition 1.2, the ascending chain condition on prime t-ideals implies that each t-prime of R is branched. Hence we can apply Theorem 1.10. □

Recall that a domain R has *finite character* (respectively, *t-finite character*) if each nonzero element of R belongs to at most finitely many maximal (respectively, t-maximal) ideals. A domain with finite character such that each nonzero prime ideal is contained in a unique maximal ideal was called by E. Matlis an *h-local domain*. Following [1], we say that R is a *weakly Matlis domain* if R has t-finite character and each t-prime ideal is contained in a unique t-maximal ideal.

Theorem 1.12. *Let R be a PvMD and consider the following conditions:*

(i) *R is a weakly Matlis domain;*
(ii) *R has t-finite character;*
(iii) *R has the $t\#\#$-property;*
(iv) *R is a $tRTP$-domain.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

If in addition each t-prime ideal of R is contained in a unique t-maximal ideal, all these conditions are equivalent.

Proof. $(i) \Rightarrow (ii)$ by definition.

$(ii) \Rightarrow (iii)$. If R has t-finite character, for each $\Lambda \subseteq t\text{-Max}(R)$ the multiplicative system of ideals $\mathcal{F}(\Lambda) := \{I\,;\, I \nsubseteq M, \text{for each}\, M \in \Lambda\}$ is finitely generated [14, Proposition 2.7]. We conclude that R is a $t\#\#$-domain by applying [15, Proposition 2.8].

$(iii) \Rightarrow (iv)$ By Theorem 1.10 (1).

Now assume that each t-prime of R is contained in a unique t-maximal ideal. Then clearly conditions (i) and (ii) are equivalent.

$(iv) \Rightarrow (ii)$ By Theorem 1.8, for each nonzero nonunit $x \in R$, the ideal xR has finitely many minimal (t)-primes. Since each t-prime is contained in a unique t-maximal ideal, then x is contained in finitely many t-maximal ideals. □

When R is a Prüfer domain, for $d = t$, from Theorem 1.11 we get [24, Theorem 2.7] and from Theorem 1.12 we get [36, Proposition 3.4].

Remark 1.13. The hypothesis that R is a PvMD in Theorems 1.11 and 1.12 cannot be relaxed. In fact each Noetherian domain is a $t\#\#$-domain [15, Proposition 2.4], but it is not necessarily a $tRTP$-domain [24, Corollary 2.2].

A *strongly discrete valuation domain* is a valuation domain such that each nonzero prime ideal is not idempotent [10, p. 145] and a *strongly discrete Prüfer domain* is a domain whose localizations at nonzero prime ideals are strongly discrete valuation domains; equivalently a domain such that $P \neq P^2$ for each nonzero prime ideal P [10, Proposition 5.3.5]. We say that a $PvMD$ R is *strongly discrete* if R_P is a strongly discrete valuation domain for each t-prime ideal P of R; equivalently, if $(P^2)_t \neq P$, for each $P \in t\text{-Spec}(R)$ [9, Lemma 3.4]. *Generalized Krull domains* were introduced by the first author in [6] and can be defined as strongly discrete PvMDs satisfying the ascending

chain condition on radical t-ideals [6, Theorem 3.5 and Lemma 3.7]. In the Prüfer case, that is for $d = t$, this class of domains coincides with the class of *generalized Dedekind domains* introduced by N. Popescu in [40]. A Krull domain is a generalized Krull domain of t-dimension one [6, Theorem 3.11].

Theorem 1.14. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a generalized Krull domain;*
(ii) *R is a strongly discrete PvMD satisfying the $t\#\#$-property;*
(iii) *R is a strongly discrete PvMD satisfying the $tRTP$-property.*

Proof. $(i) \Rightarrow (ii)$ follows from Theorem 1.11 and $(ii) \Rightarrow (iii)$ follows from Theorem 1.10(1).

$(iii) \Rightarrow (i)$. By Theorem 1.8, R is a strongly discrete PvMD such that each proper v-finite divisorial ideal has finitely many minimal primes. Hence R is a generalized Krull domain by [6, Theorem 3.9]. □

In the Prüfer case, for $d = t$, we recover from Theorem 1.14 a well known characterization of generalized Dedekind domains, see for example [10, Theorem 5.5.4].

Remark 1.15. Any w-divisorial domain is a $t\#$-domain. In fact, clearly all the t-maximal ideals of a w-divisorial domain are divisorial; hence we can apply [15, Theorem 1.2].

If R is a domain such that $R_{\mathcal{F}(\Lambda)} := \bigcap_{P\in\Lambda} R_P$ is w-divisorial, for each set Λ of pairwise incomparable t-primes, then $\mathcal{F}(\Lambda)$ is v-finite by [9, Proposition 2.2]. Thus $t\text{-Max}(R_{\mathcal{F}(\Lambda)}) = \{P_{\mathcal{F}(\Lambda)}\,;\, P \in \Lambda\}$ [9, Lemma 2.1]. It follows that, given two different sets Λ_1 and Λ_2 of pairwise incomparable t-primes of R, we have $R_{\mathcal{F}(\Lambda_1)} \neq R_{\mathcal{F}(\Lambda_2)}$. Therefore R is a $t\#\#$-domain.

Conversely, it is not true that a $t\#\#$-domain is w-divisorial. In fact each Noetherian domain has the $t\#\#$-property [15, Proposition 2.4], but a w-divisorial Noetherian domain must have t-dimension one [9, Theorem 4.2].

An integral domain R is an *almost Krull domain* if R_M is a rank-one discrete valuation domain for each t-maximal ideal M of R. Almost Krull domains were studied by Kang under the name of t-almost Dedekind domains in [30, Section IV]. A Krull domain is an almost Krull domain with t-finite character. In dimension one, the class of almost Krull domains coincides with the class of almost Dedekind domains introduced by R. Gilmer [17]. Gilmer showed that an almost Dedekind domain satisfying the #-property must be Dedekind [19, Theorem 3]. Next, we extend this result to almost Krull domains. First, we give the following characterization of almost Krull domains, which follows directly from the definitions.

Lemma 1.16. *Let R be an integral domain. Then R is an almost Krull domain if and only if R is a strongly discrete PvMD of t-dimension one.*

Theorem 1.17. *Let R be an integral domain. Then the following conditions are equivalent:*

(i) *R is an almost Krull domain satisfying the $t\#$-property;*
(ii) *R is an almost Krull domain satisfying the $t\#\#$-property;*
(iii) *R is a Krull domain.*

Proof. $(i) \Rightarrow (ii)$ Since an almost Krull domain has t-dimension one (Lemma 1.16).

$(ii) \Rightarrow (iii)$ By Lemma 1.16 and Theorem 1.14, if (ii) holds, R is a generalized Krull domain of t-dimension 1. Hence R is a Krull domain by [6, Theorem 3.11].

$(iii) \Rightarrow (i)$ Follows from Theorem 1.14. □

We end this Section by putting into evidence that the $t\#\#$-property and the $tRTP$-property of a PvMD are related respectively to the $\#\#$-property and the RTP-property of its t-Nagata ring.

Theorem 1.18. *Let R be a PvMD. Then:*

(1) *R is a t#-domain if and only if $R\langle X\rangle$ is a #-domain.*
(2) *R is a t##-domain if and only if $R\langle X\rangle$ is a ##-domain.*
(3) *R is a $tRTP$-domain if and only if $R\langle X\rangle$ is a RTP-domain.*

Proof. (1) follows from [15, Theorem 3.6]. The proof of (2) is similar and it is obtained by using Proposition 1.3 and the characterization of Prüfer ##-domains and PvMDs satisfying the t##-property proved respectively in [20, Theorem 3] and [15, Proposition 2.8 (7)].

(3) Follows from Proposition 1.3,Theorem 1.8 (v) and Corollary 1.9 (iv). □

2. w-Divisoriality and w-stability

The notion of w-divisoriality has been studied in [9]. A domain R is an integrally closed w-divisorial domain if and only if it is a weakly Matlis PvMD such that each t-maximal ideal is t-invertible [9, Theorem 3.3] and R is an integrally closed domain such that each t-linked overring is w-divisorial if and only if it is a weakly Matlis strongly discrete PvMD, equivalently R is a w-divisorial generalized Krull domain [9, Theorem 3.5]. We now introduce the notion of w-stability and show that in PvMDs w-divisoriality and w-stability are strictly related; thus extending some results proved by B. Olberding for Prüfer domains.

As before, if T is a t-linked overring of R, we denote by w the star operation induced on T by the w-operation on R and by w' the w-operation on T. We say that a w-ideal I of R is *w-stable* if I is w-invertible in the (t-linked) overring $E(I) := (I : I)$, that is if $(I(E(I) : I))_w = E(I)$, and we say that R is *w-stable* if each w-ideal of R is w-stable.

Our first result is a generalization of [39, Theorems 3.3 and 3.5] and shows in particular that the study of w-divisorial domains can be reduced to the t-local case. We recall that a valuation domain is stable if and only if it is strongly discrete [10, Proposition 5.3.8].

Lemma 2.1. *Let R be a quasi-local domain. Then a nonzero ideal I of R is stable if and only if $I^2 = xI$ for some $x \in I$.*

Proof. This follows from [39, Lemma 3.1] and [10, Lemma 7.3.4]. □

Theorem 2.2. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is w-stable;*
(ii) *Each w-ideal I of R is divisorial in $E(I)$;*
(iii) *R has t-finite character and R_M is stable for each t-maximal ideal M of R.*

Proof. $(i) \Rightarrow (ii)$. Let I be a w-ideal of R. Denote by v' the v-operation on $E(I)$. Since I is w-stable, we have $E(I) = (I(E(I) : I))_w \subseteq (I_{v'}(E(I) : I))_w \subseteq E(I)$. Hence $(I_{v'}(E(I) : I))_w = E(I)$ and $I = IE(I) = (I(E(I) : I)I_{v'})_w = ((I(E(I) : I))_w I_{v'})_w = I_{v'}$.

$(ii) \Rightarrow (i)$. Let I be a w-ideal of R and set $J = (E(I) : I)$. Proceeding like in the proof of [39, Theorem 3.5 ((2) $\Rightarrow$ (1))], we have $(E(I) : IJ) = E(I)$ and hence $E((IJ)_w) = E(I)$. Thus $(IJ)_w$ is a divisorial ideal of $E(I)$. It follows that $(IJ)_w = (E(I) : (E(I) : IJ)) = E(I) : E(I) = E(I)$, that is I is a w-stable ideal.

$(i) \Rightarrow (iii)$ Let M be a t-maximal ideal of R and let $I = JR_M$ be a nonzero ideal of R_M, where J is an ideal of R which can be assumed to be a w-ideal (since $J_wR_M = JR_M$). By w-stability in R, $(J(E(J) : J))_w = E(J)$; in particular, $J(E(J) : J)R_M = E(J)R_M$. Since $1 \in E(J)R_M = J(E(J) : J)R_M \subseteq I(E(I) : I) \subseteq E(I)$, then $I(E(I) : I) = E(I)$. Hence I is a stable ideal of R_M and therefore R_M is stable.

We next show that R has t-finite character. Let M be a t-maximal ideal of R. Since R_M is a quasi-local stable domain, then $M^2R_M = mMR_M$ for some $m \in M$ (Lemma 2.1). Set $I(M) :=$

$mR_M \bigcap R$. The ideal $I(M)$ is a t-ideal of R and $M^2 \subseteq I(M)$. Hence M is the only t-maximal ideal of R containing $I(M)$. From this, and by checking t-locally, we get $(I(M) : I(M)) = R$. Since R is w-stable, $I(M)$ is a w-invertible ideal of R. Thus $I(M)$ is divisorial.

Now, let $\{M_\alpha\}$ be a family of t-maximal ideals of R such that $\bigcap M_\alpha \neq (0)$. We want to show that $\{M_\alpha\}$ is a finite family.

Set $I_\alpha := I(M_\alpha)$ and let $J_\alpha := (\Sigma_{\beta\neq\alpha}(R : I_\beta))_w$. Note that J_α is a fractional ideal of R since $\bigcap_{\beta\neq\alpha} I_\beta \supseteq \bigcap_{\beta\neq\alpha} M_\beta^2 \neq 0$. We claim that $(J_\alpha : J_\alpha) = R$. To show this, we prove that $(J_\alpha : J_\alpha) \subseteq R_M$ for each t-maximal ideal M of R. Let $x \in (J_\alpha : J_\alpha)$. We first assume that $M \notin \{M_\beta\}_{\beta\neq\alpha}$. Since M_β is the only t-maximal ideal of R containing I_β, then $(R : I_\beta) \subseteq R_M$ for each $\beta \neq \alpha$. Hence $x \in xJ_\alpha \subseteq J_\alpha \subseteq R_M$. If $M = M_\gamma$ for some $\gamma \neq \alpha$. We have $x(R : I_\gamma) \subseteq (\Sigma_{\beta\neq\alpha}(R : I_\beta))_w$, and since I_γ is w-invertible, then $x \in (\Sigma_{\beta\neq\alpha}(I_\gamma(R : I_\beta))_w)_w$. Moreover, for $\beta = \gamma$, we have $(I_\gamma(R : I_\gamma))_w = R$, and for $\beta \neq \gamma$, $(R : I_\beta) \subseteq R_{M_\gamma}$. Hence $(\Sigma_{\beta\neq\alpha}(I_\gamma(R : I_\beta))_w)_w \subseteq R_{M_\gamma}$. Thus $x \in R_M$, which prove the claim.

Now, for each α, set $T_\alpha := \bigcap_{\beta\neq\alpha} M_\beta$. We claim that $T_\alpha \nsubseteq N$ for each t-maximal ideal $N \notin \{M_\beta\}_{\beta\neq\alpha}$. By the w-stability, J_α is a w-invertible ideal of $(J_\alpha : J_\alpha) = R$. In particular, J_α is w-finite. Thus $(R : J_\alpha)_N = (R_N : J_{\alpha N}) = R_N$ (since $I_\beta \nsubseteq N$ for each $\beta \neq \alpha$). On the other hand, we have $(R : J_\alpha)_N = (R : \Sigma_{\beta\neq\alpha}(R : I_\beta))R_N = (\bigcap_{\beta\neq\alpha} I_\beta)R_N$. Thus $\bigcap_{\beta\neq\alpha} I_\beta \nsubseteq N$. Since $\bigcap_{\beta\neq\alpha} I_\beta \subseteq T_\alpha$, then $T_\alpha \nsubseteq N$, in particular, $T_\alpha \nsubseteq M_\alpha$ for each α.

Now we proceed as in the proof of [25, Theorem 3.1]. Set $T := \Sigma T_\alpha$. By the above result T is not contained in any t-maximal ideal of R, hence $T_t = R$. Thus $(\Sigma_{i=1}^n T_i)_t = R$ for some finite subset $\{T_1, \dots, T_n\}$ of $\{T_\alpha\}$. Let $\{M_1, \dots, M_n\}$ be the corresponding set of t-maximal ideals. If $M_\alpha \notin \{M_1, \dots, M_n\}$ for some α, then $\Sigma_{i=1}^n T_i \subseteq M_\alpha$, which is impossible. Hence $\{M_\alpha\}$ is finite. Therefore R has t-finite character.

$(iii) \Rightarrow (i)$ Let I be a w-ideal of R and let $M_1, \dots, M_n$ be the t-maximal ideals of R containing I. Since I is t-locally stable then $IR_{M_i} = J_iE(I_{M_i})$ for some finitely generated ideal $J_i \subseteq I$, $i = 1, \dots, n$. Choose $y \in I$ such that $y \notin M$ for each t-maximal ideal $M \neq M_i$ containing the ideal $H := \Sigma J_i$ and consider the ideal $J := H + Ry$ of R. Clearly J is finitely generated. One can easily check that $IR_N = JE(I_N)$ for each t-maximal ideal N of R. We next show that $E(I_N) = E(I)_N$ for each t-maximal ideal N of R. Let $x \in E(I_N)$. Since $I_N = JE(I_N)$, then $xJ \subseteq I_N$. Hence $sxJ \subseteq I$ for some $s \in R \setminus N$. Let M be a t-maximal ideal of R. Then $sxI_M = sxJE(I_M) \subseteq IE(I_M) \subseteq I_M$. Thus $sxI_M \subseteq I_M$ for each t-maximal ideal M of R, so that $sxI \subseteq I$. Hence $x \in E(I)_N$. It follows that $E(I_N) = E(I)_N$, for each t-maximal ideal N, and $I = \bigcap_N I_N = \bigcap_N JE(I_N) = \bigcap_N JE(I)_N = (JE(I))_w$.

Finally, $(I(E(I) : I))_N = I_N(E(I) : JE(I))_N = I_N(E(I)_N : JE(I)_N) = I_N(E(I_N) : I_N) = E(I_N) = E(I)_N$, for each t-maximal ideal N. Therefore $(I(E(I) : I))_w = E(I)$ and so I is a w-stable ideal of R. □

Proposition 2.3. *Let R be a w-stable domain. Then:*

(1) *Each t-maximal ideal of R is divisorial.*
(2) *t-Spec(R) is treed.*
(3) *R satisfies the ascending chain condition on prime t-ideals.*

Proof. (1) Let M be a t-maximal ideal of R. If M is not divisorial, then $M_v = R$. Thus $E(M) = (R : M) = R$ and M is t-invertible. Hence M is divisorial, which is impossible.

(2) and (3) follow from Theorem 2.2 because a quasi-local stable domains has these properties [39, Theorem 4.11]. □

The previous proposition shows that w-stable domains have some properties in common with generalized Krull domains [6]. We now prove that w-stability of t-primes enforces a PvMD to be a generalized Krull domain. For Prüfer domains, this follows from [13, Theorem 5] or [36, Theorem 4.7].

Theorem 2.4. *Let R be a PvMD. The following conditions are equivalent:*

(i) *R is a generalized Krull domain;*
(ii) *Each radical t-ideal of R is divisorial and each divisorial ideal is w-stable;*
(iii) *Each radical t-ideal of R is w-stable;*
(iv) *Each t-prime ideal of R is w-stable.*

Proof. We shall freely use Proposition 1.5.

(i) ⇒ *(ii)*. Since t-Spec(R) is treed and a t-ideal of R has finitely many minimal primes, a radical t-ideal of R is a t-product of finitely many t-primes [7, Lemma 2.5]. Hence each radical t-ideal of R is divisorial [7, Proposition 3.1].

Let I be a divisorial ideal of R. If I is t-invertible, hence w-invertible, then $E(I) = R$ and so I is w-stable. If I is not t-invertible, then consider the ideal $H := (I(R : I))_w$. By [7, Proposition 2.6], we have $H = (P_1 \cdots P_n)_w$, where $n \geq 1$ and each P_i is a strong t-prime. Thus $E(P_i) = (R : P_i) \subseteq (R : H) = (R : I(R : I)) = ((R : (R : I)) : I) = E(I)$. Since P_i is t-maximal in $E(P_i)$ (Proposition 1.6) and $E(P_i)$ is t-linked over R then P_i is t-invertible in $E(P_i)$ by [6, Corollaries 3.2 and 3.6]. Hence $(P_i(E(P_i) : P_i))_w = (P_i(E(P_i) : P_i))_t = (P_i(E(P_i) : P_i))_{t'} = E(P_i)$, where t' denotes the t-operation on $E(P_i)$. Thus $P_iE(I)$ is w-invertible in $E(I)$, for each i. It follows that H is w-invertible in $E(I)$ and so I has the same property.

(ii) ⇒ *(iii)* ⇒ *(iv)* are clear.

(iv) ⇒ *(i)*. Let P be a t-prime of R. Since P is w-invertible in $E(P)$, then $P \neq (P^2)_w$ and so $P \neq (P^2)_t$. Thus R is a strongly discrete PvMD.

To prove that R is a generalized Krull domain, it is enough to show that R has the $t\#\#$-property (Theorem 1.14). Let T be a t-linked overring of R and denote by t' the t-operation on T. Let M be a t'-maximal ideal of T. Since T is a PvMD, then $T = E(M)$. The ideal $P = M \bigcap R$ is a t-prime of R and $M = (PT)_{t'} = (PT)_w$ (cf. [31, Proposition 2.10] and [6, Proposition 2.4]). Thus $R \subseteq E(P) \subseteq E(M) = T$. Since P is w-stable, then PT is w-invertible in T, and hence M is w-invertible in T. So, M is a t'-invertible t'-ideal of T (since $w = t'$ in T). In particular M is a divisorial ideal of T. We conclude that T is a $t\#$-domain by applying [15, Theorem 1.2]. □

It is known that a generalized Dedekind domain need not be stable [13, Example 10]. In fact an integrally closed domain is stable if and only if it is a strongly discrete Prüfer domain with finite character [36, Theorem 4.6]. Hence a generalized Dedekind domain is stable if and only if it has finite character. We now extend these results to generalized Krull domains.

Lemma 2.5. *A domain with t-finite character is a strongly discrete PvMD if and only if it is a generalized Krull domain.*

Proof. A strongly discrete PvMD is a generalized Krull domain if and only if each nonzero nonunit has finitely many minimal t-primes [6, Theorem 3.9]. We conclude by recalling that in a PvMD two incomparable t-primes are t-comaximal. □

Theorem 2.6. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is integrally closed and w-stable;*
(ii) *R is a w-stable PvMD;*
(iii) *R is a strongly discrete PvMD with t-finite character;*

(iv) *R is a generalized Krull domain with t-finite character;*
(v) *R is a w-stable generalized Krull domain;*
(vi) *R is a PvMD with t-finite character and each t-prime ideal of R is w-stable;*
(vii) *R is w-stable and each t-maximal ideal of R is t-invertible.*

Proof. $(i) \Rightarrow (ii)$. If M is a t-maximal ideal of R, then R_M is an integrally closed stable domain by Theorem 2.2. Hence R_M is a valuation domain [36, Theorem 4.6].

$(ii) \Rightarrow (iii)$. For each t-maximal ideal M of R, R_M is a valuation stable domain (Theorem 2.2). Hence R_M is a strongly discrete valuation domain [10, Proposition 5.3.8]. The t-finite character follows again from Theorem 2.2.

$(iii) \Leftrightarrow (iv)$ by Lemma 2.5.

$(iv) \Rightarrow (v)$ R_M is stable, for each $M \in t\text{-Max}(R)$, because it is a strongly discrete valuation domain [10, Proposition 5.3.8]. By the t-finite character, R is w-stable (Theorem 2.2).

$(v) \Rightarrow (vii)$ because each t-maximal ideal of a generalized Krull domain is t-invertible [6, Corollary 3.6].

$(vii) \Rightarrow (i)$ By Theorem 2.2, R_M is a local stable domain, for each $M \in t\text{-Max}(R)$. Since M is t-invertible, MR_M is a principal ideal. Hence R_M is a valuation domain [39, Lemma 4.5] and R is integrally closed.

$(iv) \Leftrightarrow (vi)$ follows from Theorem 2.4. □

By Theorem 2.4, each divisorial ideal of a generalized Krull domain is w-stable. Hence a w-divisorial generalized Krull domain is w-stable. Several characterizations of w-divisorial generalized Krull domains were given in [9, Theorem 3.5]. The following theorem says something more in terms of w-stability; similar results for Prüfer domains were obtained by Olberding [36, 38].

Theorem 2.7. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is an integrally closed w-divisorial w-stable domain;*
(ii) *R is a w-stable w-divisorial PvMD;*
(iii) *R is a w-divisorial generalized Krull domain;*
(iv) *R is a weakly Matlis w-stable PvMD;*
(v) *R is a weakly Matlis strongly discrete PvMD;*
(vi) *R is a weakly Matlis generalized Krull domain.*

Proof. $(i) \Leftrightarrow (ii)$ by Theorem 2.6.

$(ii) \Rightarrow (iv)$ because a w-divisorial domain is weakly Matlis [9, Theorem 1.5].

$(iii) \Leftrightarrow (v)$ by [9, Theorem 3.5].

$(v) \Leftrightarrow (vi)$ by Lemma 2.5.

$(iii) + (vi) \Rightarrow (i)$ and $(iv) \Leftrightarrow (v)$ by Theorem 2.6, because a weakly Matlis domain has t-finite character. □

From Theorems 2.6 and 2.7, we immediately get:

Corollary 2.8. *Let R be an integrally closed w-stable domain. Then R is w-divisorial if and only if each nonzero t-prime ideal is contained in a unique t-maximal ideal.*

A domain is stable and divisorial if and only if it is totally divisorial [38, Theorem 3.12]. The following is the t-analogue of this result in the integrally closed case.

Corollary 2.9. *Let R be an integral domain. The following conditions are equivalent:*

(i) *R is a w-stable w-divisorial PvMD;*
(ii) *R is integrally closed and each t-linked overring of R is w-divisorial.*

Proof. By [9, Theorem 3.5], if R is integrally closed, each t-linked overring of R is w-divisorial if and only if R is a weakly Matlis strongly discrete PvMD. We conclude by applying Theorem 2.7. □

We recall that each overring of a domain R is t-linked if and only if $d = w$ on R [4, Theorem 2.6] and that each overring of a stable domain is stable [39, Theorem 5.1]. We now prove that w-stability is preserved by t-linked extension.

Theorem 2.10. *Let R be an integral domain and T a t-linked overring of R. If R is w-stable then T is w'-stable, where w' denotes the w-operation on T.*

Proof. We shall use Theorem 2.2. Since $R \subseteq T$ is t-linked, for each t'-maximal ideal M of T, there is a t-maximal ideal N of R such that $R_N \subseteq T_M$ [4, Proposition 2.1]. Hence T_M is an overring of a stable domain and is therefore stable [39, Theorem 5.1].

We next show that T has t-finite character. Let N be a t-maximal ideal of R and let $\{M_\alpha\}$ be a family of t-maximal ideals of T such that $\bigcap_\alpha M_\alpha \neq (0)$ and $M_\alpha \cap R \subseteq N$ for each α. Set $S := \bigcap_\alpha T_{M_\alpha}$. Then S is a stable domain since it is an overring of the stable domain R_N [39, Theorem 5.1]. The prime ideals $P_\alpha = M_\alpha T_{M_\alpha} \cap S$ of S are pairwise incomparable, since $S_{P_\alpha} = T_{M_\alpha}$ for each α. We have $(0) \neq \bigcap_\alpha M_\alpha \subseteq \bigcap_\alpha P_\alpha$, and, since S is treed [39, Theorem 4.11 (ii)] and has finite character [39, Theorem 3.3], then $\{P_\alpha\}$ must be finite. Hence $\{M_\alpha\}$ is also a finite set. Since R has t-finite character, it follows that T has t-finite character. □

We do not know whether the integral closure of a w-stable domain is w'-stable. In fact the integral closure of a domain R is not always t-linked over R [5, Section 4] and we cannot apply Theorem 2.10. However, the w-integral closure $R^{[w]} := \bigcup\{(J_w : J_w)\,; J$ a finitely generated ideal of $R\}$ is integrally closed and t-linked over R [4, Proposition 2.2 (a)]. Thus we immediately get:

Corollary 2.11. *The w-integral closure of a w-stable domain is a w'-stable PvMD.*

We end by remarking that, in the integrally closed case, w-divisoriality and w-stability correspond to divisoriality and stability of the t-Nagata ring. We shall make use of Proposition 1.3.

Theorem 2.12. *Let R be an integral domain. Then:*

(1) *R has t-finite character if and only if $R\langle X\rangle$ has finite character.*
(2) *R is a Weakly Matlis PvMD if and only if $R\langle X\rangle$ is an h-local Prüfer domain.*
(3) *R is a strongly discrete PvMD if and only if $R\langle X\rangle$ is a strongly discrete Prüfer domain.*
(4) *R is a generalized Krull domain if and only if $R\langle X\rangle$ is a generalized Dedekind domain.*
(5) *R is an integrally closed w-divisorial domain if and only if $R\langle X\rangle$ is an integrally closed divisorial domain.*
(6) *R is an integrally closed w-stable domain if and only if $R\langle X\rangle$ is an integrally closed stable domain.*

Proof. Denote by $c(f)$ the content of a polynomial $f(X) \in R[X]$.

(1) We have $\mathrm{Max}(R\langle X\rangle) = \{MR\langle X\rangle; M \in t\text{-}\mathrm{Max}(R)\}$. Since $f(X) \in MR[X]$ if and only if $c(f)_v \subseteq M$, if R has t-finite character, then $R\langle X\rangle$ has t-finite character. The converse is clear.

(2) Follows from (1) and Proposition 1.3.

(3) For $M \in t\text{-}\mathrm{Max}(R)$, we have that $R\langle X\rangle_{MR\langle X\rangle} = R[X]_{MR[X]} = R_M(X)$ is a strongly discrete valuation domain if and only if R_M has the same property.

(4) Follows from (3) and Proposition 1.3 by recalling the definitions.

(5) When R is integrally closed, R is divisorial if and only if R is an h-local Prüfer domain such that each maximal ideal is invertible [23, Theorem 5.1] and R is w-divisorial if and only if R is a weakly Matlis PvMD such that each t-maximal ideal is t-invertible [9, Theorem 3.3]. Hence we can

conclude by applying part (2) and recalling that, for each $M \in t\text{-Max}(R)$, $MR\langle X\rangle$ is invertible if and only if M is t-invertible [30, Theorem 2.4].

(6) Follows from [36, Theorem 4.6], Theorem 2.6 and statements (1) and (3). □

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